## Input-Output and Hybrid LCA (Subject Editor: Sangwon Suh)

# Power Series Expansion and Structural Analysis for Life Cycle Assessment

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**Abstract**

**Goal, Scope and Background.** The usefulness of power series expansion for an LCA system has often been doubted, as those systems may not possess the unique properties that enable power series expansion and analyses based on the power series. This paper surveys the existing literature on power series expansion of monetary input-output system and discusses how the power series expansion can be utilized for more general systems including the LCA model.

**Methods.** The inherent properties of matrices that are capable of producing power series forms for their inverse and, further, can utilize structural path analysis are analyzed. Using these analyses, the way how a matrix that is not eligible for structural analyses is converted into an eligible form is investigated. A numerical example is presented to demonstrate the findings.

**Results.** The necessary and sufficient condition for an indecomposable, real square technology matrix can be expressed using power series was identified. Two additional conditions for a technology matrix to be utilized for structural analyses using power series expansion are discussed as well. It was also shown that an LCA system that fulfills the Hawkins-Simon condition can be easily converted into the form that is eligible for structural analysis by rescaling the columns and rows.

**Discussion.** As a numerical example, an application of accumulative structural path analysis for an LCA system is shown. The implications of the results are discussed in a more plain language as well.

**Conclusions.** The survey presented in this paper provides not only the conditions under which a linear system is expressed using a power series form but also the way to appropriately convert a system to utilize the rich analytical tools using power series expansion for structural analyses.

**Recommendations and Perspectives.** Widely used LCA databases and software tools have employed the linear systems approach as the basis. Much of these developments in the domain of LCA have been made, however, in isolation of the rich findings of IOA. There will be much to benefit LCA through an active dialogue between the two disciplines.
There are rich analytical tools available through the use of power series expansion. The current survey will help software developers and LCA practitioners to apply such tools in LCA.

**Keywords:** Input-output analysis; IOA; LCA practitioners; LCA software developers; monetary input-output system; power series expansion; structural path analysis

### Introduction

Doing an LCA study means working with numbers. The ecoinvent inventory data contains a few million numbers, and a typical impact assessment methods like Impact2002 contains many thousands additional numbers. A bunch of numbers as such, however, makes no sense. They must be combined in the right way, using elementary operations like addition, multiplication, and division. The few sources that address the issue of combining the data in the right way are based on a formulation in terms of linear systems; see, e.g., Heijungs & Suh (2002) and Suh & Huppes (2005).

As academic disciplines proliferate in the course of their developments, it is often the case that seemingly remote theories in different disciplines in fact share much in common. Such recognition sometimes opens up the possibilities of gaining insights by using the findings from other disciplines or, sometimes, even of integrating them. The use of linear systems in economics, ecology and industrial ecology is an example that different disciplines share a common formal expression in representing their systems. Linear systems have been proposed, developed and used not only in LCA (Heijungs 1994), but in the fields of Input-Output Analysis (IOA; see Leontief 1936) and Ecological Network Analysis (ENA; see Hannon 1973) as well. But these developments have taken place in a rather isolated fashion. More recently there have been a number of efforts to utilize or merge the systems across the disciplines, using the common mathematical ground as a basis (Joshi 1999, Suh 2004a, Suh 2004b, Suh 2005). The current paper concerns the linear system that is a generalization of those used in these fields its focus being on the conditions of power series expansion that enables various analyses.

The use of power series expansion was introduced in the early history of IOA, where it was originally devised to solve the computational problem in inverting a large matrix under the lack of computational capacity (Waugh 1950). The power series form of a Leontief system has been extensively studied in relation to economic stability issues notably by Solow (1952). Although rapid development of computer technology quickly diminished the utility of power series expansion as a computational tool, its useful characteristics are still of analytical and computational interests (see e.g., Treloar 1997, Lave et al. 1995, Weber and Schnabl 1998, Lenzen 2001, Suh 2004a, Peters 2007). For instance, Treloar (1997) applied the power series expansion to extract important paths of the Australian inter-industry linkages in terms

of their embodied energy, and Lave et al. (1995) and Lenzen (2001) used it to quantify the truncation error of process-based LCA. Outside the Leontief input-output domain, Patten (1982) implicitly applied the power series form to analyze nutrient and energy flows between ecosystem components; Suh (2004a) used it to identify important suppliers in terms of their environmental impacts in a hybrid LCA setting. The virtue of using system expansion form is that it unravels a complex network relationship enabling a detailed insight on the structure of the system.

Nevertheless, there have been confusions and misunderstandings on power series expansion especially for the systems other than the monetary Leontief input-output system, as those systems do not necessarily possess the unique properties of Leontief input-output system that enables power series expansion and analytical meaningfulness of the results. For instance, Frischkecht & Kolm (1995) and Schmidt (1995) applied the power series expansion form for LCA matrices, while their physical meaning is very doubtful (see Heijungs & Suh 2002, pp. 104–105). As it can be seen later in this paper, such a problem arises as physical systems, such as LCA and ENA models, may not share some of the key properties of Leontief system that are related to power series form of an inverse. The same holds for Physical Input-Output Tables (PIOTs) and mixed-unit input-output tables where certain conditions are not met with these systems (Kratterl and Kratena 1990, Kratena et al. 1992, Pedersen 1999, Stahmer et al. 2003, Hoekstra 2003, Suh 2004b). Even within the monetary IOA, this situation may occur, for instance for heavily subsidized sectors (see footnote 1).

This paper discusses how the power series expansion form can be utilized for a general system that includes LCA, mixed-unit input-output, PIOT and ecological network models. We discuss the conditions for a generalized linear system to have a power series form for its inverse, and, more importantly, the way to appropriately convert a system to utilize the rich analytical tools using power series expansion for structural analyses. Many LCA software tools and databases including some of the largest commercial packages have long been using linear systems for their embedded computational structures. The linear systems used in these tools may not be eligible for structural analyses using power series form due to e.g. the tradition of treating waste flows. The current survey helps gaining insights on why some systems are not eligible for structural analyses and on how to manipulate such system to utilize the structural analyses.

Many of the proofs or elements of proofs presented in this article are already made or implied by previous studies over the last century, e.g. Frobenius (1908), Oldenburger (1940), Waugh (1950), Solow (1952), Debreu and Herstein (1953) and Fiedler and Pták (1962), but we felt it necessary to bring these rich findings from various other fields together, and, thus, they are included in this paper. The intention is to draw an attention to these findings, which are scattered over a long period of time, in a mathematically consistent way, rather than to develop original proofs. This then provides a basis for the development of new techniques in LCA, and for the incorporation of techniques that are already available to specialists in IOA, ENA, or any other applied science that is based on linear systems, but that is not yet known to the LCA community.

This paper is structured as follows. Section 1 discusses the power expansion for IOA, and Section 2 discusses more general systems, with an emphasis on LCA. Sections 3, 4 and 5 show how the various results of Section 1 can be applied to LCA. An LCA application, using accumulative structural path analysis, in Section 6, demonstrates how techniques that were already available for IOA can now be employed to LCA systems. Section 7 concludes. At parts, the discussion will be rather technical, involving concepts from linear algebra like matrix norms, eigenvalues, and the Jordan canonical form. Readers that are not familiar with such concepts are advised to move to the application, in Section 6. Or interested readers may find it useful to read a less technical overview of linear systems used in LCA by e.g., Suh and Huppes (2005).

## 1 Conditions for a Leontief System

It was Waugh (1950) who, to our knowledge, first proved that a Leontief system always has an inverse that can be expressed as a power series. Let the direct requirement matrix be given as the square matrix **B**. The $i$–$j$th element of a direct requirement matrix **B**, that is $b_{ij}$, in an open Leontief system shows the amount of industry output from industry $i$ directly required by industry $j$ to produce one unit of output in monetary term. Waugh (1950) showed that the Leontief inverse $(\mathrm{I} - \mathrm{B})^{-1}$can be expressed using a power series as

$$(\mathbf{I} - \mathbf{B})^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \ldots \tag{1}$$

provided that

$$\lim_{m\to\infty} \mathbf{B}^m = \mathbf{0}. \tag{2}$$

Here we reproduce a proof following Chiang (1984, pp. 120–121). The equality in (1) is proved if

$$(\mathbf{I} - \mathbf{B})\,(\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 \cdots + \mathbf{B}^m) = \mathbf{I}, \tag{3}$$

where $m \to \infty$ is assumed. The Left-Hand-Side (LHS) of (3) can be rewritten as

$$\begin{aligned}&(\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots + \mathbf{B}^m) - \mathbf{B}\,(\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots + \mathbf{B}^m)\\ &= (\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots + \mathbf{B}^m) - (\mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots + \mathbf{B}^{m+1}),\end{aligned} \tag{4}$$

which becomes $\mathbf{I} - \mathbf{B}^{m+1}$. Therefore, the condition that (3) holds if (2) holds.

Waugh (1950) proved that a monetary Leontief system always fulfills the condition shown in (2) using the norm of **B**. Let us define the norm of **B** as $N$(**B**), which here is the largest column sum of non-negative **B** (Bowker 1947, Waugh 1950): $N(\mathbf{B}) = \max_j (\sum_i b_{ij})$.

In general, a monetary Leontief system, representing a feasible economy, $N(\mathbf{B}) < 1$, as, for each industry, the sum of all purchases from other industries to produce one unit of its output is generally less than unity given positive labor and profit.[1] One of the properties of a norm is that the product of the norms of two matrices is always larger than or at most equal to the norm of the product of the two matrices. For the same matrix, $\mathbf{B}$, such property implies $[N(\mathbf{B})]^2 \geq N(\mathbf{B}^2)$. As $N(\mathbf{B}) < 1$ for non-negative $\mathbf{B}$ of a feasible economy, $[N(\mathbf{B})]^m$ always approaches 0 as $m$ approaches positive infinity, which necessitates $N(\mathbf{B}^m)$ approaching 0. Hence, by the definition of a norm, all elements of $\mathbf{B}^m$ become 0 as $m$ approaches positive infinity and thus the condition (2) always holds for any Leontief input-output system (Miller & Blair 1985).[2]

Solow (1952) provides a sufficient condition for a Leontief system to fulfill the condition in (2): all eigenvalues of a nonnegative, indecomposable Leontief system $\mathbf{B}$ must have a modulus less than unity.[3] There again, the condition that all column sums of $\mathbf{B}$ being less than unity is used to prove the theorem.

It should be noted that the condition for all column sums of a Leontief direct requirement matrix being less than unity is a sufficient condition, not a necessary one. Furthermore, such a condition is not necessarily satisfied in the physical and mixed-unit systems such as the ones used for LCA, mixed-unit input-output tables and physical input-output tables (PIOT).

## 2 Power Series Expansion for a more General System

Linear systems are used for various fields outside the monetary input-output domain. For instance, LCA uses linear systems to describe the commodity or functional flows between processes (see e.g., Heijungs 1994, Schmidt & Schorb 1995, Heijungs & Suh 2002, Suh 2004a), and many LCA databases and software tools including ecoinvent, Simapro 6 and CMLCA are employing such technique as a computational algorithm embedded in the database or the software tools.

Mixed-unit input-output tables provide another example where linear system is employed. Konijn et al. (1997) and Hoekstra (2003) employed physical units for a number of metal flows in an input-output table, while other flows are noted in the traditional, monetary values. By using physical units, the input-output system can be free from the price inhomogeneity and fluctuation, better representing physical inter-relationships between industries. Another linear system that deals with physical flows is the physical input-output table, PIOT. It shows the relationships between industries in terms of the mass flows between them (Kratterl & Kratena 1990, Kratena et al. 1992, Pedersen 1999, Stahmer et al. 2003). Hubacek and Giljum (2003) utilize PIOT for analyzing the appropriation of land due to European exports. Suh (2004b) borrowed the computational structure of LCA in dealing with the problem of representing waste flows in PIOT.

Linear systems started to be used for ecosystems since 1970s (Hannon 1973). In ecology linear systems are used to describe the structure of an ecosystem in terms of the energy and nutrient flows, and currently the world largest ecosystem network databases are based on the same, linear framework (Patten 1982, Szyrmer & Ulanowicz 1987, Christensen & Pauly 1992). Baily (2000) utilized the ecological network models for industrial systems and Suh (2005) generalized economic input-output model and ecological networks model in a unified framework.

Although the system description in these applications looks very similar to the traditional Leontief input-output model, there are several notable differences. First, the most intuitive difference is that, unlike the single unit monetary input-output tables, the column sum is not relevant in mixed-unit systems, as it adds, for instance, kg with MJ. Second, as a consequence, the norm of $(\mathbf{I} - \mathbf{A})$ is often bigger than 1, so that the sufficient condition for power series expansion in Miller and Blair (1985) and Solow (1952) is not necessarily satisfied in these systems. Third, some of these systems do not belong to the class of M-Matrices and some principal minors can be negative, while they still produce a non-negative inverse (see Heijungs & Suh (2002), p. 17).[4] Fourth, LCA technology matrices are, at least as they are currently used, not necessarily used in a normalized form. Finally, the expression $\mathrm{I} - \mathrm{A}$ has only physical meaning when the coefficients on the diagonal of $\mathbf{A}$ are dimensionless, because $1 - a_{ii}$ contains for every $i$ a dimensionless 1, and the algebra of units permits only the addition of quantities with the same unit. Thus these diagonal elements may be \$/\$, kg/kg, or MJ/MJ, but never \$/kg, kg/MJ, etc.

To illustrate these differences, consider, for instance, a simple, fictitious LCA system, consisting of a technology matrix $\mathbf{A}$ (2×2) and an intervention matrix $\mathbf{E}$ (1×2), as shown in **Table 1**.[5]

**Table 1:** Arbitrary LCA system, comprising 2 unit processes

| | Power generation | Coal mining |
|---|---|---|
| Electricity (kWh) | 1500 | –3000 |
| Coal (Ton) | –0.01 | 0.1 |
| $CO_2$ emission (kg) | 20 | 10 |

[1] In reality it may be the case that some of the column sums of a technology matrix are larger than unity due to various reasons including that the capital investments are endogenized within the intermediate part, the sector actually made loss in that fiscal year or/and the sector is heavily relying on subsidies.

[2] We used the more direct proof by Miller and Blair (1985) instead of the one by Waugh (1950).

[3] Solow (1950) distinguishes decomposable and indecomposable matrices and provides a proof for each, and argues that the previous proof by Brauer (1947) is false, as it omits the decomposability condition. As the proof for a decomposable matrix is a directly corollary of the other, and as, in reality, it is hardly possible to have a decomposable matrix for the systems considered in this paper, we limit ourselves to indecomposable matrices throughout the paper, while the proof for the decomposable system is left to the readers.

[4] Fiedler and Pták (1965) provide well-known equivalent definitions of M-Matrices and associated proofs, and Poole and Boullion (1974) offer a broader spectrum of M-Matrices with references. For example, a square matrix with non-positive off-diagonal elements with a non-negative inverse is an M-matrix.

[5] Note that in this paper we followed the notation used in LCA, where the matrix **A** stands for what is generally noted as (**I** – **A**) in input-output analysis. To avoid any confusion, we used **B** for the direct requirement matrix of input-output accounts in this paper; see (1). Note also that use of **A** in this way is not common in input-output literature but is nothing new either (see e.g., Hawkins & Simon 1949).

Table 1 shows that producing 1500 kWh of electricity requires 0.01 ton of Coal, while coal mining produces 0.1 ton of coal by using 3000 kWh. An LCA technology matrix can be directly extracted from Table 1 such that

$$\mathbf{A} = \begin{bmatrix} 1500 & -3000 \\ -0.01 & 0.1 \end{bmatrix}. \tag{5}$$

The amount of total $CO_2$ emission is calculated by $\mathbf{EA}^{-1}$ = [0.0175 625], which shows that 1 kWh of electricity and 1ton of Coal generates 0.0175 kg and 625 kg of $CO_2$ emission, respectively. This system works fine in producing its correct LCA result.

However, although the system in (5) still generates correct, positive inverse[6], its power series form, that is $\mathbf{I} + (\mathbf{I} - \mathbf{A}) + (\mathbf{I} - \mathbf{A})^2 + (\mathbf{I} - \mathbf{A})^3 + \cdots$, does not converge and thus all the useful analytical tools that make use of power series expansion form cannot be used for the system above. What follows are a discussion, on what conditions a more general system has a power series form for its inverse, and whether there is a way to convert a system that is not eligible for a power series expansion like the one above to the one that has a power series form that can be used for various analyses.

## 3 Conditions of Power Series Expansion for a Generalized Linear System

What follows is a rather technical elaboration on the conditions for a more general system to have a power series expansion form for its inverse. These proofs will serve as a technical basis for subsequent discussions to follow. Non-technical readers may skip this section.

**Lemma 1:** The inverse of an indecomposable, real square matrix **A** can be expressed as $\mathbf{I} + (\mathbf{I} - \mathbf{A}) + (\mathbf{I} - \mathbf{A})^2 + \cdots$ if and only if all eigenvalues of $(\mathbf{I} - \mathbf{A})$ have a modulus less than unity.

**Proof:** Consider an $n \times n$ real matrix **A**. From (1) and (2), and by substituting $\mathbf{A} = \mathbf{I} - \mathbf{B}$, it follows that the inverse of a matrix **A** can be expressed with a power series

$$\mathbf{A}^{-1} = \mathbf{I} + (\mathbf{I} - \mathbf{A}) + (\mathbf{I} - \mathbf{A})^2 + (\mathbf{I} - \mathbf{A})^3 + \cdots \tag{6}$$

if and only if $(\mathbf{I} - \mathbf{A})$ to the power $m$ converges to a null matrix when $m$ approaches infinity. An eigenvalue $\mu_i$ of **B** is defined as a possibly complex value that satisfies $\mathbf{Bx}_i = \mu_i \mathbf{x}_i$. By locating eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ to each column of **X**, $\mathbf{BX} = \mathbf{X}\hat{\boldsymbol{\mu}}$ holds, where is a square matrix in which $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ are on its diagonal and 0 on elsewhere. If **X** is non-singular, it follows that $\mathbf{X}^{-1}\mathbf{BX} = \hat{\boldsymbol{\mu}}$. By raising to the $m$th power

$$(\mathbf{X}^{-1}\mathbf{BX})^m = \underbrace{(\mathbf{X}^{-1}\mathbf{BX})(\mathbf{X}^{-1}\mathbf{BX})\cdots(\mathbf{X}^{-1}\mathbf{BX})}_{m} = \mathbf{X}^{-1}\mathbf{B}^m\mathbf{X} = \hat{\boldsymbol{\mu}}^m, \tag{7}$$

[6] The issue of positive inverse has been a point of discussion in input-output literatures especially in relation to the make and use framework and so called commodity assumptions (ten Raa et al. 1984, ten Raa 1988, Kop Jansen & ten Raa 1990, Steenge 1990, Konijn 1994, Londero 1999, Almon 2001). Negative elements in an inverse are better accepted in LCA and interpreted as the amount of inputs 'substituted' by non-primary multiple outputs (see Heijungs & Suh 2002, pp. 45*ff*, Kagawa & Suh, forthcoming, cf. Stone et al. 1963).

and, thus, the matrix $\mathbf{B}^m$ has the eigenvalues $\mu_1{}^m, \mu_2{}^m, \mu_3{}^m, \ldots, \mu_n{}^m$. Since both $\mathbf{X}^{-1}$ and **X** are independent of $m$, $\mathbf{B}^m$ converges to **0** as $m$ increases if and only if each of the $\mu_i{}^m$ converges to 0. Therefore, $\mathbf{B}^m$ converges to a null matrix if and only if $|\mu_i| < 1$ for all $i$.

When **B** has less than $n$ number of linearly independent eigenvectors, **X** is not invertible and equation (7) does not hold. In that case, the Jordan canonical form can be utilized to complete the proof. For any **B** there exist a non-singular **C** that satisfies $\mathbf{CBC}^{-1} = \mathbf{J}$, where **J** is the Jordan canonical form. **J** has the following block diagonal form,

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}_k \end{bmatrix} = \begin{bmatrix} \begin{matrix} \mu_1 & 1 & 0 & \cdots & 0 \\ 0 & \mu_1 & 1 & \cdots & 0 \\ 0 & 0 & \mu_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} \mu_k & 1 & 0 & \cdots & 0 \\ 0 & \mu_k & 1 & \cdots & 0 \\ 0 & 0 & \mu_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_k \end{matrix} \end{bmatrix} \tag{8}$$

where $\mu_1, \mu_2, \ldots, \mu_k$, are the eigenvalues of **B**, and all non-diagonal blocks are zero; the dimension of each Jordan block corresponds to the multiplicity of the eigenvector it represents. By using the same technique used in (7), it follows that $\mathbf{CB}^m\mathbf{C}^{-1} = \mathbf{J}^m$. For $m \geq j$, $m$th power of any $j \times j$ Jordan block, $\mathbf{J}_i^m$ is expressed by

$$\mathbf{J}_i^m = \begin{bmatrix} \mu_i^m & {}^mC_1\mu_i^{m-1} & {}^mC_2\mu_i^{m-2} & \cdots & {}^mC_{m-j+1}\mu_i^{m-j+1} \\ 0 & \mu_i^m & {}^mC_1\mu_i^{m-1} & \cdots & {}^mC_{m-j}\mu_i^{m-j} \\ 0 & 0 & \mu_i^m & \cdots & {}^mC_{m-j-1}\mu_i^{m-j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_i^m \end{bmatrix}, \tag{9}$$

where the combinatorial factor ${}^nC_r = n! \,/\, n!\,(n-r)!$. By taking a limit, $\lim_{m\to\infty} \mathbf{J}_i^m$ will vanish to a null matrix if and only if $|\mu_i| < 1$, and, thus, $\mathbf{J}^m$ will be **0** as $\to\infty$ if and only if $|\mu_i| < 1$ for all $i$.

**Corollary 1:** All real eigenvalues of a real, square matrix **A** that fulfils the condition in lemma 1 lies within the open interval (0, 2).

**Proof:** $\mathbf{Bx} = (\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{x} - \mathbf{Ax} = \mu\mathbf{x} \Leftrightarrow \mathbf{Ax} = (1 - \mu)\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (1 - \mu) = \lambda$ and, as $|\mu_i| < 1$ for all $i$ if **A** satisfies the condition in lemma 1, all real eigenvalues of **A** lies within the open interval (0, 2).

**Corollary 2:** For a non-negative, real **B** the condition in lemma 1 is reduced to have the largest real eigenvalue of **B** within the open interval (0, 1).

**Proof:** The Frobenius theorem adds a more powerful constraint on the condition for a non-negative matrix, **B**. As the eigenvalue with largest modulus of non-negative **B** is real,

positive and simple (Frobenius 1908, p. 471), the necessary and sufficient condition for a non-negative, real **B** to be a null matrix by raising power of $m$, where $m \to \infty$ is that the largest real eigenvalue of **B** lies within the open interval (0, 1).

**Lemma 2:** Let **A** be a square real invertible matrix, where **B** = **I** – **A** is non-negative. Then, the following statements are equivalent:

a) $(\mathbf{B})^m = \mathbf{0}$, as $m \to \infty$.
b) $r_\sigma(\mathbf{B}) < 1$, where $r_\sigma$ represents the spectral radius, i.e. the modulus of the largest eigenvalue
c) $N(\mathbf{B}^m) \to 0$
d) $\mathbf{A}^{-1}$ exists and is equal to $\mathrm{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots$

Moreover, a sufficient condition for any of these four statements is $N(\mathbf{B}) < 1$.

**Proof:** See the previous proofs, Atkinson (1989, p. 490 *ff.*) and Isaacson and Keller (1966, p. 14 *ff.* and 63).

Lemmas 1 and 2 are well known and have been extensively discussed in input-ouptut literatures (cf. Hadamard's and McKenzie's theorems of dominant diagonal, see Takayama 1974). Furthermore, these conditions hold for any arbitrary systems including LCA, PIOT, mixed-unit input-output system and ecological network models with a square matrix. However, it should be noted that the lemmas present merely the condition for a system to have a proper power series expansion form for its inverse and does not tell us whether the power series form conveys analytical meaning to be used in, for instance, the Structural Path Analysis (SPA) and the Environ analysis. Such an issue will be dealt with in the following section.

### 4 Analytical Meaningfulness of Power Series Expansion for a more General System

In a previous section it was shown that the system in (3) does not have a power series expansion form for its inverse, although the system correctly provides a solution. In fact, the eigenvalues of **A** are 1500 and 0.1, and therefore does not comply with the conditions of lemmas 1 and 2.

By manipulating units, however, for instance kWh to MWh, (5) can be modified to have eigenvalues within the range specified in lemma 1 without changing the result. Consider a downscaled system of

$$\mathbf{A} = \begin{bmatrix} 1.5 & -3 \\ -0.01 & 0.1 \end{bmatrix}. \qquad (10)$$

The total $CO_2$ emissions calculated by using (9) are 17.5 kg and 625 kg per 1 MWh of electricity and 1 ton of coal, respectively, which are identical to the previous result. System (10) has norm bigger than 1 for (**I** – **A**), namely 3.9, which disqualifies the sufficient condition of lemma 2, but has eigenvalues for (**I** – **A**) –0.521 and 0.921, and, therefore, fulfils the condition (b) of lemma 2. A general applicable type of rescaling is proposed by Janiszowksi (2003): the matrix **A** is replaced by $\alpha\mathbf{A}$ with $\alpha$ chosen in such a way that the eigenvalues of $\alpha\mathbf{A}$ are within the allowed range, e.g. $\alpha = 1/|\lambda_{\max}|$, and the obtained inverse is multiplied by $\alpha$. Thus, it is possible to use a power series expansion for finding the inverse of a general square matrix by introducing a suitable rescaling.

Now the question is whether the rescaled system in (10) can be used for structural analyses. Unfortunately applying **A** to the formula (6) gives meaningless figures instead of round-by-round requirements used in e.g., SPA. For instance, consider the third term,

$$(\mathbf{I}-\mathbf{A})^2 = \begin{bmatrix} -0.5 & 3 \\ 0.01 & 0.9 \end{bmatrix} \begin{bmatrix} -0.5 & 3 \\ 0.01 & 0.9 \end{bmatrix} = \begin{bmatrix} 0.28 & 1.2 \\ 0.004 & 0.84 \end{bmatrix}. \qquad (11)$$

This operation gives no information on upstream requirements even though the power series form apparently converges to $\mathbf{A}^{-1}$.[7]

In fact, the system requires one more condition in order to be utilized for round-by-round structural analysis: each column of **A** needs to be normalized by its diagonal element so that each column represents the input and output relationship per one unit of its output located in the diagonal. For instance, the system (10) will correctly give the upstream requirements when each column is normalized by its diagonal such that:

$$\mathbf{A} = \begin{bmatrix} 1.5 & -3 \\ -0.01 & 0.1 \end{bmatrix} \begin{bmatrix} 1/1.5 & 0 \\ 0 & 1/0.1 \end{bmatrix} = \begin{bmatrix} 1 & -30 \\ -0.0067 & 1 \end{bmatrix}. \qquad (12)$$

Note that changing units was done by rescaling rows, while the operation in (12) rescales columns (see also Heijungs & Suh 2002. p. 109). Then, the second order of upstream requirement is calculated by

$$(\mathbf{I}-\mathbf{A})^2 = \begin{bmatrix} 0 & 30 \\ 0.0067 & 0 \end{bmatrix} \begin{bmatrix} 0 & 30 \\ 0.0067 & 0 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}. \qquad (13)$$

If interpreted, 1 MWh of power generation requires 0.0067 ton of coal to produce one unit of its own commodity, and the 0.0067 ton of coal required by power generation, in turn, requires electricity, 30×0.0067 = 0.2 MWh. Therefore, in order to compute round by round requirements using a power series, the matrix should be manipulated in such a way that (net) output figure is located in its diagonal, and each column is normalised by its (gross) output figures. In case there is no self-consumption as is in many LCA systems, this condition means 1 for all diagonal elements.[8] In addition, most applications will require non-positive off-

[7] For instance, using a plain flow diagram, the first order of upstream requirement on coal by electricity can be calculated by 0.01×3/0.1 = 0.3 (ton coal/1.5 kWh electricity), while the RHS of eq. (11) indicates 0.004 (element in 2, 1).

[8] The self-consumption part appears in an aggregated statistics such as national input-output tables. Unlike the common misunderstandings, this part does not represents the in-facility consumption of outputs such as the electricity used by the power plant that produces it, as what is generally counted is only the net output. These self-consumption parts in most cases show the inputs from other facilities that *happen to be* classified in the same industry category of the receiving facility. An example is food processing sector, where, for instance, a corn mill, classified as a food processing industry, produces starch and corn oil and these products are used by another food processing company for refinery and then by another to finally produce consumer products. Such a statistical artifact is less likely to appear in LCA, where the process resolution is much higher.

diagonal, which assures that there will be only one commodity produced by each process.[9] In other words, the system has to have been properly *allocated*[10]. Even if this additional condition is not fulfilled, the round-by-round calculation can still be used, but its interpretation needs caution. In this case, each round-by-round calculation will show the input requirements by that round minus the amount of input *saved* by producing the co-product that is assumed to be a by-product. In case the amount saved is larger than the amount used in that round, negative values will appear, which can be interpreted as the net amount of input saved by producing the co-product in that round. For a more detailed discussion that involves positive off-diagonal elements, see Lee (1982).

The example shown above demonstrates that a system that fulfills the condition in Lemma 1 can be converted into the form that is eligible for structural analysis using power series form for its inverse by rescaling its columns and rows. Then a natural question is whether such a rescaling of rows and columns alters the result or not. Let us consider a basic Life Cycle Inventory (LCI) problem, $\mathbf{M} = \mathbf{E}\mathbf{A}^{-1}\hat{\mathbf{y}}$, where **E** is environmental intervention per certain accounting period or operation time, **A** is a technology matrix based on the same accounting period or operation time, **y** contains the functional unit and **M** shows the inventory results (Heijungs & Suh 2002). Let us substitute **A** and **y** by $\hat{\mathbf{r}}\mathbf{A}$ and $\hat{\mathbf{r}}\mathbf{y}$, respectively, so that all rows are arbitrarily rescaled. Then the RHS of the equation becomes $\mathbf{E}\mathbf{A}^{-1}\hat{\mathbf{r}}^{-1}\hat{\mathbf{r}}\hat{\mathbf{y}} = \mathbf{E}\mathbf{A}^{-1}\hat{\mathbf{y}} = \mathbf{M}$ (see Heijungs & Suh 2002), so that the original solution is preserved (for a seminal work, see Fisher 1965). Similarly, it can be shown that each column can be rescaled without altering the results. Let us substitute **A** and **E** by rescaled matrices, $\mathbf{A}\hat{\mathbf{c}}$ and $\mathbf{E}\hat{\mathbf{c}}$, respectively. Then the RHS of the given equation becomes $\mathbf{E}\hat{\mathbf{c}}\hat{\mathbf{c}}^{-1}\mathbf{A}^{-1}\hat{\mathbf{y}} = \mathbf{E}\mathbf{A}^{-1}\hat{\mathbf{y}} = \mathbf{M}$, so that the result, **M** is always preserved after the arbitrary rescaling of rows and columns. On top of that, it may be necessary to perform additional operations on **A**, for instance changing the order of rows or columns. A general formulation is as follows: with transformations given by $\mathbf{A}' = \mathbf{RAC}$, $\mathbf{E}' = \mathbf{EC}$, and $\mathbf{y}' = \mathbf{Ry}$, the problem of finding a power expansion of $\mathbf{A}^{-1}$ thus reduces into the problem of determining matrices **R** and/or **C**, such that $\mathbf{A}'$satisfies the conditions specified in Lemma 2.

This series of discussion leads to the following :

**Corollary 3:** The necessary and sufficient condition for a real, invertible **A** to be used for structural analyses using its power series form, $\mathbf{A}^{-1} = \mathbf{I} + (\mathbf{I} - \mathbf{A}) + (\mathbf{I} - \mathbf{A})^2 + (\mathbf{I} - \mathbf{A})^3 + \cdots$, is to satisfy all of the following three conditions:

(a) $a_{ii} = 1$,
(b) $a_{ij} \leq 0$ for all $i$, $j$ ($i \neq j$), and
(c) all eigenvalues are within the unity in modulus for $\mathbf{B} = (\mathbf{I} - \mathbf{A})$

The conditions (a) and (b) imply that the system is monofunctional, is arranged such that all production figures are located in its diagonal and that each column is related to the gross production value of the column.[11] Thus, a general recipe for constructing **R** and **C** can be made as follows:

(1) first, appropriate steps must be made to transfer by-products to other sectors, or to split sectors that produce more than one product in several sectors. Waste outputs should be changed into waste treatment inputs by changing the sign of an entire row of **A** and **y**.
(2) we make **R** a permutation matrix, a matrix with a 1 in every row and column and zeros at all other places. The places where the 1 is to occur is determined by the discrepancy between the sector and 'its' product.
(3) we make **C** a rescaling matrix that transforms the diagonal of **A** into numbers between 0 and 1.

In general, the make and use framework applied to a general linear system works this out (Kagawa and Suh, forthcoming).[12]

## 5 Relationship with Hawkins-Simon Condition

In this section we discuss the relationship between the conditions for power series expansion of an inverse and the conditions for having positive inverse matrix. The Hawkins-Simon (H-S) condition states that a necessary and sufficient condition for a Leontief system **B** to have a positive inverse is that all principal minors[13] of $(\mathbf{I} - \mathbf{B})$ be positive (Hawkins & Simon 1949).

Let us generalize the H-S condition for an arbitrary system.

**Definition:** If all principle minors of a real square matrix are positive, then the matrix is said to fulfill the generalized Hawkins-Simon condition.

Note that, unlike the Leontief system, there exist some **A** that fulfils the generalized H-S condition, of which inverse, $\mathbf{A}^{-1}$ is non-positive.[14] However any **A** that fulfils the conditions (a) and (b) in Corollary 3 will always have positive solution (see, Fiedler and Pták (1962)). Further, the general relationship between the condition in lemma 1 and the generalized H-S condition can be easily derived.

**Corollary 4:** Any square, real **A** that fulfils the condition in lemma 1 also fulfils the generalized Hawkins-Simon condition.

**Proof:** The H-S condition is satisfied if and only if all real eigenvalues of **A** are positive, and, from the Corollary 1, all real eigenvalues of **A** that fulfils the condition in lemma 1 lies within (0, 2).[15]

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[9] Having non-positive off-diagonal in a technology matrix implicitly applies so called 'by-product technology' model (see e.g., Konijn 1994).

[10] Allocation refers in LCA to the act of manipulating a technology matrix such that every process produces only one product. See Heijungs & Suh (2002) for more details and references.

[11] In practice, column or row permutation matrices can be used to relocate the production figures to the diagonal. In case there should be a self-consumption portion due to e.g., poor statistical resolution, diagonals can be less than one but should still indicate the amount of net production per unit of gross production of itself (see also footnote 8).

[12] Models used under the make and use framework including commodity-technology, industry-technology and mixed technology models can be used for this purpose. However, for a mixed unit systems, certain restriction applies: Industry-technology model requires a single unit for the part that the model is applied. Commodity-technology model does not impose such restriction. For an application of make and use framework for mixed-unit LCA system, see Suh and Huppes (2002). For a seminal review of make and use framework, see Konijn (1994).

[13] For the definition of a principal minor, we refer to Meyer and Mayer (2001).

[14] For example, the absolute form of **A** in (7), fulfills H-S condition, while its inverse is non-positive.

[15] The equivalence between the generalized Hawkins-Simon condition and all real eigenvalues being positive follows easily as the determinant of a matrix is the product of all eigenvalues of the matrix and the product of non-real eigenvalues of a real matrix is positive.

**Lemma 3:** For any real, square **A** that fulfils the conditions (a) and (b) of Corollary 3, the condition (c) is equivalent to the generalized H-S condition.

**Proof:** As any **A** that fulfils the conditions in Corollary 3 will always have positive solution (see, Fiedler and Pták (1962)), it is obvious that the generalized H-S condition is necessary condition for any **A** that fulfils (a) and (b) of Corollary 3 to be used for structural analyses. It remains to prove that Hawkins-Simon condition is a sufficient condition for any **A** that fulfils (a) and (b) of Corollary 3 to be used for structural analyses. For any **A** that fulfils (a) and (b) of Corollary 3 and a positive scalar $c$, a non-negative **B** exists such that $\mathbf{A} = c\mathbf{I} - \mathbf{B}$. Since, there exist a non-negative real eigenvalue $\mu_{max}$ of non-negative **B** such that $\mu_{max} \geq |\mu|$ for all eigenvalues $\mu$ of **B**, and $\mu = c - \lambda$, where $\lambda$ is the eigenvalue of **A**, $\mu_{max}$ is determined by $\lambda_{min}$ as $\mu_{max} = c - \lambda_{min}$, where »$_{min}$ is an eigenvalue of **A** such that $\lambda_{min} \leq \lambda$ for all $\lambda$. Since $\lambda_{min}$ is positive for all **A** that fulfils the generalized H-S condition[16], $|\mu| < c$ if **A** satisfies the generalized H-S condition. Since $a_{ii} \leq 1$ for all **A** that satisfies (a) in Corollary 3, without any loss of generality one can set $c = 1$. Then, $|\mu| < 1$ for all eigenvalues $\mu$ of **B**. Hence, by the definition of generalized H-S condition, any A that fulfils the conditions (a) and (b) of Corollary 3 that fulfils the generalized H-S condition also satisfies the condition (c) and *vice versa*.

By the definition of generalized H-S condition and lemma 3, the lower bound of det(**A**) for **A** that satisfies the conditions of Corollary 3 is given. Furthermore, the upper bound of det(**A**) for **A** that fulfils the conditions of Corollary 3 can easily follow.

**Corollary 5:** $0 < \det(\mathbf{A}) \leq 1$ for any **A** that fulfils the conditions in Corollary 3.

**Proof:** Since $a_{ii} > 0$ and $a_{ij} \leq 0$ $(i \neq j)$ for all **A** that fulfils the conditions of Corollary 3, triangulizing **A** always adds non-positive values to each diagonal, whereas the result of triangulization will need to show all positive diagonal in order to have all positive principal minors. Therefore, each diagonal element of the triangulized **A** is equal or less than 1. Since $\det(\mathbf{A}) = (t_1 \cdot t_2 \cdot t_3 \cdots t_n)$, where $t_1, t_2, \ldots t_n$ are the diagonal elements of the triangulized **A**, determinant of **A** that fulfils the conditions of Corollary 3 is less than unity.

## 6 Application

In this section, we demonstrate how power series expansion and corresponding analytical tools can be used for generalized linear systems by means of a numerical example. The analytical tool that we chose is Structural Path Analysis (SPA) and Accumulative Structural Path Analysis (ASPA) based on Suh (2001). In analyzing a system it is often necessary to decompose the complex network system into individual paths. In this case the power series form of an inverse can be used to gain insights. Here we draw an example as an LCA problem, however, the methodology generally holds for other mixed-unit and physical flow models. Let us consider an LCA problem and define $\mathbf{B} = (\mathbf{I} - \mathbf{A})$, then (6) becomes

$$\mathbf{A}^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots \tag{14}$$

For the given environmental matrix, **E**, the amount of environmental intervention per unit of output by each process is calculated by

$$\mathbf{E}\,(\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots), \tag{15}$$

and especially the amount of environmental intervention $i$ due to a unit output of process $j$ is decomposed in scalar by

$$e_{ij} + \sum_l e_{il} b_{lj} + \sum_l \sum_m e_{il} b_{lm} b_{mj} + \cdots, \tag{16}$$

for all $i, l, m, \ldots$ Note that each permutation of indices, $l, m\ldots$ in each term in (16) represents a specific input path required to deliver one unit of $j$ to outside the system. A second-order of upstream path, $\sum_i e_{i1} b_{14} b_{43}$, for example, shows the amount of the $i$th environmental intervention generated by the first commodity to produce fourth commodity to deliver one unit of the third commodity, which is set for the functional unit.

With complete decomposition of an inverse shown in (16), SPA allows us to locate the key paths throughout the system that contribute significant environmental impacts. The most important paths from product $p$ to process $q$ with regard to the environmental intervention $i$ is defined here as

$$e_{ip} b_{pq} = \max_{l,j}(e_{il} b_{lj}), \tag{17}$$

for all $l, j$, which means that the flow of product $p$ to process $q$ generates environmental intervention $i$ the most among the single order paths. Similarly, the most important $n$th order path, ( $\overbrace{p, q, r, \cdots x}^{n-1}$ ) can be defined as

$$e_{ip} \overbrace{b_{pq} b_{qr} \cdots b_{wx}}^{n-1} = \max_{j,l,m,\ldots,o} (e_{il} \overbrace{b_{lj} b_{jm} \cdots b_{no}}^{n-1}), \tag{18}$$

for all indices $j, l, m, \ldots, o$. The most important path in each order represents the most important packet of supply-chain at given length. In general, as the length of the supply-chain increases its impact decreases, since the 'contents' of far-upstream inputs in the final products are generally smaller than that of near-upstream inputs (Defourny & Thorbecke 1984, Treloar 1997, Lenzen 2004)[17].

While SPA concerns only the direct generation of environmental intervention connected with each individual commodity path, ASPA takes both direct and indirect impacts into account. It identifies the bottleneck of a supply-chain where major environmental problems are caused by the particular commodity flow and the upstream supply-chain from that on.

As an alternative to (16), the amount of environmental intervention $i$ associated with a unit output of commodity $j$ is decomposed in more compact form of scalar quantities by

$$e_{ij} + \sum_l e_{il} \tilde{a}_{lj}, \tag{19}$$

where $\tilde{a}_{lj}$ is $i$-$j$th element of $\tilde{\mathbf{A}} = \mathbf{A}^{-1}$. Or by

[16] See the proof of Corollary 4 and Fiedler and Pták (1962) p. 385.

[17] A seminal review and analysis of available techniques can be found in Lenzen (2003).

$$e_{ij} + \sum_l e_{il} b_{lj} + \sum_l \sum_m e_{il} b_{lm} \tilde{a}_{mj} \,, \tag{20}$$

which is now expanded up to the third term. Likewise the equation (20) can be further expanded infinite times. Note that $e_{il}\tilde{a}_{lj}$ in (19) shows the direct and indirect generation of environmental intervention $i$ stemming from the first order input path of $l$ to the $j$th process including all upstream interactions originated from the particular input.

In general SPA uses the certain rows and columns of the power series form in (6) and ASPA uses the power series form in (6) and the inverse, $\mathbf{A}^{-1}$. For both analyses, it is obvious that the inverse needs to be expressed using power series form and, furthermore, the power series form needs to have the analytical meaning that we would like to utilize.

Now we would like to use ASPA to analyze an LCA system. We utilized the ETH96 database that has been widely used for LCA studies (Frischknecht et al. 1996). ETH96 distinguishes more than a thousand processes and several hundreds environmental interventions, focusing mainly on European product systems. It generally corresponds to the LCA computational structure by Heijungs (1994) and Heijungs & Suh (2002) that uses mixed-units. The technology matrix of ETH96 is transformed so that it fulfils the conditions of Corollary 3 as described in the previous sections, which involves assigning output values on the diagonals and normalizing each column by its diagonal. Beside, waste flows had to be modified as they are currently marked as outputs violating the condition (b) of Corollary 3. Therefore, the signs of all waste flows have been changed, so that the flow stands for 'waste treatment service' instead of the physical waste flows. The ASPA is carried out for the product, 'Electricity, low voltage in UCPTE'. For the sake of simplicity we used $CO_2$ emission data as an indicator, while other indicators could have been be used as well. The result is shown in **Table 2**. Only those flows contribute more than 1% of the total are presented up to the third tier of upstream.

Note that some of the commercial LCA software tools provide a tree-like display, where the thickness of the arrows between the processes represents tier-wise accumulative contribution. In the existence of loops, i.e., when an output of a process requires itself through inputs from other processes, mathematical formulation of tier-wise accumulative contribution is not simple. For that, existing software tools with matrix approach utilize perturbation method to approximate tier-wise accumulative contribution, which requires additional computational resources and time.[18]

The results show that 1.2% of the total $CO_2$ emission by the European low-voltage electricity mix are produced mainly for transmission within UCPTE, and almost all of the rest is made up by the electricity supplied by individual countries (1st order). As to the contribution by individual countries,

**Table 2:** Accumulative structural path analysis for ETH96 database (Electricity, low voltage in UCPTE)

| | Order of upstream inputs | | |
|---|---|---|---|
| **% to the total** | **1st** | **2nd** | **3rd** |
| 98.6% | Electricity – Mix UCPTE | | |
| 1.2% | Infra Electricity – in UCPTE | | |
| 42.0% | Electricity – Mix UCPTE | Electricity – Mix D | |
| 20.2% | Electricity – Mix UCPTE | Electricity – Mix I | |
| 9.0% | Electricity – Mix UCPTE | Electricity – Mix E | |
| 6.5% | Electricity – Mix UCPTE | Electricity – Mix NL | |
| 6.4% | Electricity – Mix UCPTE | Electricity – Mix F | |
| 5.2% | Electricity – Mix UCPTE | Electricity – Mix GR | |
| 3.7% | Electricity – Mix UCPTE | Electricity – Mix B | |
| 2.3% | Electricity – Mix UCPTE | Electricity – Mix P | |
| 1.6% | Electricity – Mix UCPTE | Electricity – Mix A | |
| 1.1% | Electricity – Mix UCPTE | Electricity – Mix Ex–YU | |
| 1.0% | Infra Electricity – in UCPTE | Transmission | |
| 20.4% | Electricity – Mix UCPTE | Electricity – Mix D | Electricity from brown coal in D |
| 16.5% | Electricity – Mix UCPTE | Electricity – Mix D | Electricity from hard coal in D |
| 10.6% | Electricity – Mix UCPTE | Electricity – Mix I | Electricity oil fired I |
| 6.5% | Electricity – Mix UCPTE | Electricity – Mix I | Electricity gas fired I |
| 5.8% | Electricity – Mix UCPTE | Electricity – Mix E | Electricity from hard coal in E |
| 4.3% | Electricity – Mix UCPTE | Electricity – Mix GR | Electricity from brown coal in GR |
| 3.4% | Electricity – Mix UCPTE | Electricity – Mix NL | Electricity gas fired NL |
| 3.4% | Electricity – Mix UCPTE | Electricity – Mix D | Electricity gas fired D |
| 3.3% | Electricity – Mix UCPTE | Electricity – Mix F | Electricity from hard coal in F |
| 2.8% | Electricity – Mix UCPTE | Electricity – Mix NL | Electricity from hard coal in NL |
| 2.7% | Electricity – Mix UCPTE | Electricity – Mix I | Electricity from hard coal in I |
| 2.1% | Electricity – Mix UCPTE | Electricity – Mix B | Electricity from hard coal in B |
| 1.9% | Electricity – Mix UCPTE | Electricity – Mix E | Electricity from brown coal in E |
| 1.7% | Electricity – Mix UCPTE | Electricity – Mix F | Electricity gas fired F |
| 1.3% | Electricity – Mix UCPTE | Electricity – Mix B | Electricity gas fired B |
| 1.2% | Electricity – Mix UCPTE | Electricity – Mix P | Electricity from hard coal in P |

UCPTE: Union for the Coordination of Production and Transmission of Electricity, W: Germany, I: Italy, E: Spain, NL: Netherlands, F: France, GR: Greece, B: Belgium, P: Poland, A: Austria, EX–YU: former Yugoslavia

electricity mix by Germany is the largest contributor accounting for 42% of the total $CO_2$ emission by average low-voltage electricity grid-mix in UCPTE, followed by that of Italy, contributing 20% (2nd order). Regarding the German average electricity, electricity from Brown coal (20%) and Hard coal (17%) combustion power plants dominates marking these two the most important third-order accumulated $CO_2$ embodiment paths. The largest $CO_2$-contributing power technology in Italy is oil-fired power plant, which is different from the case of Germany, accounting for 11% of the total (3rd order). Using ASPA method the complex electricity grid-mix of Europe can be analyzed in terms of their contribution to certain emission. This example concerns only the $CO_2$ aspect, but of course, other indicators at different aggregation level such as total global warming potential, total human toxicological impact or total weighted impact can be used as well. A standard contribution analysis (Heijungs et al. 2005) of the same dataset does not specify the different orders, but effectively boils down to an infinite order ASPA.

Likewise power series expansion and corresponding analytical tools such as ASPA can be utilized in other physical systems including hybrid-unit input-output table, hybrid IO-LCA, PIOT and ecological network system, given that they fulfill the conditions discussed in this paper.

### 7 Summary and Discussion

In this paper, we discussed the conditions for an indecomposable, real square matrix to have a power series form for its inverse in the context of structural analyses that are used in LCA, mixed-unit input-output analysis, PIOT and ecological network analysis. First, some known results for the monetary Leontief input-output system are reviewed. For the monetary input-output system, it is shown that the basic property of the direct requirement matrix **B**, namely the column sums of **B** are always less than unity, fulfills the sufficiency of having power series form for $(\mathbf{I}-\mathbf{B})^{-1}$. For the physical systems such as LCA and mixed-unit input-output matrices, however, such sufficient condition is not met and, thus, the power series form may be unavailable. The condition for a physical system to have a power series form and, more importantly, the condition for a system to be eligible for a structural analysis have been presented. The necessary and sufficient condition for an indecomposable, real square matrix **A** to have an inverse $\mathbf{A}^{-1}$ that can be expressed as $\mathbf{I}+(\mathbf{I}-\mathbf{A})+(\mathbf{I}-\mathbf{A})^2+\cdots$ was identified to be that all eigenvalues of $(\mathbf{I}-\mathbf{A})$ have the modulus less than unity. In order to utilize structural analysis using power series expansion, however, **A** needs to fulfill two additional conditions, namely $a_{ii}=1$, and $a_{ij}\leq 0$ for all $i, j$ $(i \neq j)$. It was also shown that a physical system that fulfills the H-S condition can be converted into the form that is eligible for structural analysis by rescaling the columns and rows. The relationships between different conditions are also discussed. Finally we demonstrated how a mixed-unit system can be transformed to utilize the analytical tools based on the power series expansion form using a numerical example. In the example we applied ASPA for an LCA system to identify the most important accumulated path in the supply-chain.

In a more plain language, the condition in lemma 1 that all eigenvalues of a matrix $(\mathbf{I}-\mathbf{A})$ have a modulus less than unity simply requires $(\mathbf{I}-\mathbf{A})^m$ approaches to a null matrix as $m \to \infty$ and *vice versa*. For this class of matrices, it is possible that the kind of physical interpretations used for round-by-round analysis is not given for the power series form of its inverse. The two additional requirements to be used for structural analyses, that are $a_{ii}=1$, and $a_{ij}\leq 0$ for all $i, j$ $(i \neq j)$, means that the system needs to be manipulated in such a way that each unit process, industry, compartment or ecosystem component needs to produce only one flow, which needs to be located at the diagonal, and each column needs to be normalized by its gross production. In other words, each activity distinguished needs to be mono-functional and each column needs to present the inputs and outputs per unit of its gross output. In LCA such a manipulation can be done by allocation procedure, and for mixed-unit input-output model and PIOT, it can be done by using make and use framework. The generalized Hawkins-Simon condition implies that a system is not self-exhaustible. In other words, no activity or group of activities should require more than what is produced. The relationships between these conditions show that any mono-functional, normalized system (the conditions (a) and (b) of Corollary 3) that is not self-exhaustible (definition of generalized H-S condition) is eligible for structural analyses using power series form.

The survey presented in this paper provides not only the conditions under which a linear system is expressed using a power series form but also the way to appropriately convert a system to utilize the rich analytical tools using power series expansion for structural analyses. Since Heijungs (1994), linear systems have been widely utilized in LCA as a general database architecture for Life Cycle Inventory (LCI), as a basic computational tool, and as a platform for wide range of analytical tools (see Heijungs and Suh 2002, for a seminal discussion). Widely used LCA databases and software tools such as Simapro 6, CMLCA, ecoinvent and its predecessor, ETH96 database all have employed the linear systems approach as the basis. Much of these developments in the domain of LCA have so far been made in isolation of the rich findings of IOA and ENA. As a result, the linear systems used in these tools may not be eligible for structural analyses using power series form due to e.g. the tradition of treating waste flows and the use of flow matrix instead of normalized coefficient matrix. The current survey tries to bridge the gaps between LCA on the one hand and IOA and ENA on the other hand, focusing on why some systems are not eligible for structural analyses and on how to manipulate such system to utilize the structural analyses.

It is our conviction that a unified science, in which LCA, IOA and ENA are just different implementations of the same general systems-theoretical concept, is an ideal for which the LCA community should strive, together with other disciplines of science. The subject of power series expansion elaborated here, however important, is only one example of how the rich findings of these other disciplines can strengthen the theoretical and analytical basis of LCA.

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[18] Suh (2005) presents a simple, scalar computation for the same issue in the context of *Environ* analysis used in systems ecology (Suh 2005, p. 262).

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